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Oligomorphic groups, their automorphism groups, and the complexity of their isomorphism*

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The paper establishes results following two interconnected directions.

1. Let G be a Roelcke precompact closed subgroup of the group $\mathrm{Sym}(\omega)$ of permutations of the natural numbers. Let $\mathrm{Aut}(G)$ denote the group of continuous automorphisms of G. Then $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ carries the topology of pointwise convergence for its (faithful) action on the cosets of open subgroups. Under the stronger hypothesis that G is oligomorphic, $\mathcal{N}_G/G$ is profinite, where $\mathcal{N}_G$ denotes the normaliser of G in $\mathrm{Sym}(\omega)$, and the topological group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is totally disconnected, locally compact.

2a. We provide a general method to show smoothness of the isomorphism relation for appropriate Borel classes of oligomorphic groups. We apply it to two such classes: the oligomorphic groups with no algebraicity, and the oligomorphic groups with finitely many essential subgroups up to conjugacy.

2b. Using this method we also show that if G is in such a Borel class, then $\mathrm{Aut}(G)$ is topologically isomorphic to an oligomorphic group, and $\mathrm{Out}(G)$ is profinite.

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1. Introduction

Background. A permutation group on an infinite set X is a subgroup of the group $\mathrm{Sym}(X)$ of permutations of X. Such a group is called *oligomorphic*

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(Cameron [4]) if for each $n \in \omega$, the canonical action of G on X^n has only finitely many orbits. The *closed* subgroups of $\mathrm{Sym}(X)$ are precisely the automorphism groups of structures with domain X. These are topological groups: a neighbourhood basis of 1 is given by the pointwise stabilisers of finite sets. Oligomorphic closed subgroups G of $\mathrm{Sym}(\omega)$ correspond exactly to automorphism groups of $\aleph_0$-categorical structures with domain ω, structures of crucial importance in model theory. Concretely, we have $G = \mathrm{Aut}(M_G)$ where M_G is the structure on ω that has an n-ary relation for each n-orbit of G. Two oligomorphic groups are topologically isomorphic if and only if the corresponding structures are bi-interpretable (Coquand, see [1]).

The close link between the topological study of oligomorphic groups and the model theory of $\aleph_0$-categorical structures will determine our perspective of oligomorphic groups in this paper. However, these groups have been studied from a wide variety of other angles. For instance, Cameron studied the possible growth of the number of orbits, both of n-element sets, and of n-tuples of pairwise distinct elements. This connects oligomorphic groups to the area of combinatorial enumeration. In theoretical computer science, they play a role in constraint satisfaction problems, when the templates are certain reducts of infinite homogeneous structures [3].

Results. We follow two interrelated directions. The *first direction* studies, for the first time in generality, the groups $\mathrm{Aut}(G)$ of topological automorphisms of an oligomorphic closed subgroup G of $\mathrm{Sym}(\omega)$, and its outer automorphism group $\mathrm{Out}(G)$. The group $\mathrm{Aut}(G)$ has a unique Polish topology, recalled in 2.3 below, that makes its action on G continuous ([22]; also see [20, Section 6]). The notation $H \leqslant_c \mathrm{Sym}(\omega)$ indicates that H is a closed subgroup of $\mathrm{Sym}(\omega)$.

Theorem 1.1. *Let $G \leqslant_c \mathrm{Sym}(\omega)$ be an oligomorphic group. Let $\mathcal{N}_G$ denote its normaliser in $\mathrm{Sym}(\omega)$.*

(a) $\mathrm{Inn}(G)$ *is closed in* $\mathrm{Aut}(G)$.
(b) The Polish group $\mathcal{N}_G/G$ is profinite.
(c) The Polish group $\mathrm{Out}(G)$ *is totally disconnected, locally compact (t.d.l.c.).*

This theorem is obtained through three separate results. Part (a) is proved for the wider setting of Roelcke precompact groups in 2.7. Part (b) follows from 3.4, and part (c) is obtained through 3.12. The "upper bound" on the complexity of $\mathrm{Out}(G)$ in (c) begs the question whether $\mathrm{Out}(G)$ is always profinite for an oligomorphic group G. It is easy to see from our proofs that the map $G \mapsto \mathrm{Out}(G)$ is a Borel invariant of oligomorphic groups. Our results open the door for a wider study of oligomorphic groups via this invariant, perhaps similar to the case of finite groups [31].

The *second direction* is to study the complexity, in the sense of Borel reducibility $\leqslant_B$ (see [12]), of the topological isomorphism relation between oligomorphic groups. The set of oligomorphic groups can be seen as a standard Borel space (see [17]). In [23] it was shown that their isomorphism relation is Borel reducible to a Borel equivalence relation with all classes countable; it was left open whether this equivalence relation is in fact much lower: is it Borel reducible to the identity relation on $\mathbb{R}$? Such an equivalence relation is called *smooth*.

We note that the research for classes of oligomorphic groups is part of a larger research programme started in [17]: to determine the complexity of the topological isomorphism relation on natural Borel classes of closed subgroups of $\mathrm{Sym}(\omega)$. For example, topological isomorphism on the class of topologically *finitely generated* compact subgroups of $\mathrm{Sym}(\omega)$ is smooth: such groups are given by the collection of isomorphism types of their finite quotients [15], and the set of types of such quotients can be seen as an element of a standard Borel space, when suitably encoded as a set of natural numbers. In contrast, isomorphism on the class of *all* compact subgroups of $\mathrm{Sym}(\omega)$ is Borel equivalent to the isomorphism relation between countable graphs, and hence not smooth [17, Thm. 4.3]. (We note that the compact closed subgroups of $\mathrm{Sym}(\omega)$ are, up to topological isomorphism, precisely the countably based profinite groups.)

We provide in Theorem 4.7 a sufficient criterion for when the isomorphism relation on a Borel class $\mathcal{V}$ of oligomorphic groups is smooth (in brief, we will say that $\mathcal{V}$ is smooth). The criterion says that there is a Borel and invariant property P picking out open subgroups from groups in $\mathcal{V}$, so that the filter of all open subgroups of a group G in $\mathcal{V}$ is generated by the finitely many conjugacy classes of subgroups enjoying P. We apply the criterion to two classes.

The first class consists of the automorphism groups of $\aleph_0$-categorical structures with no algebraicity. Here the property P says that the subgroup is the stabiliser of a number.

The second class consists of the oligomorphic groups with finitely many essential subgroups up to conjugacy. Here, an *almost essential* subgroup is an open subgroup U without a proper subgroup of finite index such that the conjugacy class of U generates the filter of open subgroups; to be *essential* means additionally to be of minimal depth, where the depth of an open subgroup U is the least length of a maximal chain of subgroups above U (noting that there are only finitely many subgroups of G above U). Definition 5.8 will provide the detail.

Theorem 1.2. *Let $\mathcal{C}$ be either*
(1) the class of oligomorphic groups with no algebraicity, or
(2) the class of oligomorphic groups with finitely many essential subgroups up to conjugacy.

(a) The topological isomorphism relation on $\mathcal{C}$ is smooth.

(b) If $G \in \mathcal{C}$ then $\mathrm{Out}(G)$ is profinite, and $\mathrm{Aut}(G)$ is isomorphic to an oligomorphic group.

The results are obtained by verifying the hypotheses that are needed for the general Theorem 4.7: for (1) see Lemma 5.4; for (2) see Lemma 5.10.

The groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ have somewhat more concrete descriptions when G is in the first class, i.e., $G = \mathrm{Aut}(M)$, for a structure M with no algebraicity. Let $\mathcal{E}_M$ denote the orbital structure of M_G, which is the reduct of M_G that for each n has the equivalence relation of being in the same n-orbit. The descriptions are as follows.
Suppose that $G = \mathrm{Aut}(M)$ for a structure M with no algebraicity. Then

$$\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathcal{E}_M) = \mathcal{N}_G, \text{ and } \mathrm{Out}(G) \cong \mathcal{N}_G/G.$$

For a proof see the arXiv version of the paper [21, Th. 5.6].

Our main motivation for considering the second class is the following. An oligomorphic group is called $\mathfrak{G}$*-finite* [8, Definition 3.4.2, pg. 33] if each open subgroup contains a least open subgroup of finite index. (The definition in Lascar's original paper [18] is different, but coincides with the present one under G-compactness.) We note that Evans and Hewitt [9] constructed canonical examples of oligomorphic groups that are not $\mathfrak{G}$-finite.

Proposition 5.11 shows that every $\mathfrak{G}$-finite group has an essential subgroup. So the class of $\mathfrak{G}$-finite groups that have only finitely many essential subgroups up to conjugacy is smooth. It might be possible a priori that *every* $\mathfrak{G}$-finite group is of this kind. As we will see in Section 5.3, the second class contains the class of automorphism groups of $\aleph_0$-categorical structures that have weak elimination of imaginaries, recently studied in [26]. So 1.2(b) above is new in particular for these groups. Each group in this class is $\mathfrak{G}$-finite [26].

Discussion. We do not know whether the isomorphism relation is smooth for the class of all oligomorphic groups, so we search for *dividing lines* D. The ideal situation would be that in the presence of D one has smoothness, and otherwise not; this is analogous to what happens with the dividing lines in classification theory. It would be a strong solution to the whole problem. If a Borel equivalence relation is not smooth, then the relation E_0 of almost equality of subsets of ω can be Borel reduced to it by a result of Harrington, Kechris and Louveau (see, e.g., [12, Thm. 6.3.1]). We search for the

most general condition ensuring smoothness, so that we can focus on possible constructions for a reduction of E_0 that exclude groups with that condition. Theorem 1.2 suggests that the "right" dividing line is *$\mathfrak{G}$-finite or without algebraicity*. So groups that fail both conditions would be needed to show the class of oligomorphic groups is not smooth. For this one could consider finite covers of non-$\mathfrak{G}$-finite groups, which are shown to exist in [8, Chapter 1].

Structure of the paper. In Section 2 we discuss the Polish topology on $\mathrm{Aut}(G)$, for G in the wider class of Roelcke precompact groups G. Then we show that $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$, thus leading to a Polish topology on $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. The core Section 3 provides model theoretic methods needed in the rest of the paper. We use model theory to show that $\mathcal{N}_G/G$ is profinite for any oligomorphic G, where $\mathcal{N}_G$ is the normaliser of G in $\mathrm{Sym}(\omega)$. Thereafter, we discuss how to represent open cosets of G by pairs of imaginaries, and use this in Subsection 3.4 to show that $\mathrm{Out}(G)$ is a t.d.l.c. group for oligomorphic G. Section 4 develops the general criterion that allows us to treat the two rather disparate cases in Theorem 1.2 uniformly; we expect it to have further applications to other Borel classes of oligomorphic groups. Section 5 then shows the criterion is applicable to these two classes, and gives further information particular to these classes.

Conventions 1.3. By G, H we will denote closed subgroups of $\mathrm{Sym}(\omega)$. All isomorphisms between topological groups will be topological. $\mathrm{Aut}(G)$ will denote the group of *continuous* automorphisms of G. We use the group theoretic notation $g^h = hgh^{-1}$ for the conjugate of g by h. By M we always denote a (countably infinite) $\aleph_0$-categorical structure. By "definable" we will mean "$\emptyset$-definable" unless otherwise noted.

2. Aut(G) for Roelcke precompact G

The main result of this section is that for any Roelcke precompact non-Archimedean group G, the group of inner automorphisms is closed in $\mathrm{Aut}(G)$. This needs some definitions and preliminaries.

A countably based Polish group is *non-Archimedean* if it has a neighbourhood basis of 1 consisting of open subgroups. Equivalently, it is topologically isomorphic to a closed subgroup of $\mathrm{Sym}(\omega)$. Recall that a *double coset* of a subgroup U of a group G is a set of the form UgU where $g \in G$. Roelcke precompactness [6] is a general property of topological groups: being precompact in the Roelcke uniformity. We will only consider non-Archimedean groups, in which case the definition can be phrased as follows.

Definition 2.1. A non-Archimedean group G is *Roelcke precompact* if each open subgroup U has only finitely many double cosets; equivalently, the left action of U on its own *left* cosets has only finitely many orbits.

It is well known that such a group G has only countably many open subgroups: such a subgroup contains the pointwise stabiliser V of a finite set, and hence is given as a union of the finitely many double cosets of V. Each oligomorphic group is Roelcke precompact by the following.

Fact 2.2 ([29], Theorem 2.4)**.** A closed subgroup G of $\mathrm{Sym}(\omega)$ is oligomorphic iff G is Roelcke precompact and the action of G on ω has only finitely many 1-orbits.

Recall that by $\mathrm{Aut}(G)$ we denote the group of *continuous* automorphisms of G.

Definition 2.3. Given a Roelcke precompact group G, we define a topology on $\mathrm{Aut}(G)$ by declaring as open the subgroups of the form

$$\{\Phi \in \mathrm{Aut}(G) \colon \forall i = 1, \dots, n, \Phi(A_i) = A_i\},$$

where $A_1, \dots, A_n$ are cosets of open subgroups of G.

Fact 2.4 ([22], Thm. 1.2, Thm. 2.1)**.** $\mathrm{Aut}(G)$ with the topology given by Definition 2.3 is non-Archimedean. This topology on $\mathrm{Aut}(G)$ is the least Baire topology, and the unique Polish topology that makes the action on G continuous.

[23, Section 3.3] develops a duality between oligomorphic groups G and countable structures $\mathcal{M}(G)$ called "coarse group" defined on the cosets of open subgroups of G. The meet groupoid $\mathcal{W}(G)$ is a variant of $\mathcal{M}(G)$ briefly introduced in [23, Remark 2.16], where it is noted that $\mathcal{M}(G)$ and $\mathcal{W}(G)$ are interdefinable via a fixed set of first-order formulas. The first author and Melnikov [19] elaborated on $\mathcal{W}(G)$, albeit in the setting of totally disconnected, locally compact groups G (where its domain consists of the *compact* open cosets). It will be convenient to use $\mathcal{W}(G)$ rather than $\mathcal{M}(G)$ in what follows.

Definition 2.5. Given a Roelcke precompact group G, the domain of the *meet groupoid* $\mathcal{W}(G)$ consists of the open cosets, together with $\emptyset$. The groupoid product $A \cdot B$ of open cosets is the usual product AB in the group; it is defined only if A is a left coset of a subgroup that B is a right coset of, so that the result is again a coset. The meet is the usual intersection as a binary operation.

Note that a coset A in G is a right coset of the subgroup $U = A \cdot A^{-1}$ and a left coset of $V = A^{-1} \cdot A$. Given that groupoids can be seen as categories, we say that U is the *domain* of A and that V is the *codomain* of A. Thus, $A \cdot B$ is defined iff the codomain of A coincides with the domain of B. By a *two-sided coset* of a subgroup U we mean a left coset of U that is also a right coset of U. The two-sided cosets of U are in a canonical bijection with $N_G(U)/U$, whence their number equals $[N_G(U) : U]$, the index of U in its normaliser in G.

Fact 2.6. For conjugate subgroups $U, V \in \mathcal{W}(G)$, there is a bijection from the set of two-sided cosets of U to the set of elements of $\mathcal{W}(G)$ with domain U and codomain V.

Proof. Since U and V are conjugate, there exists a coset $A \in \mathcal{W}(G)$ with domain U and codomain V. We claim that the map β given by $\beta(C) = C \cdot A$ is a bijection as required.

β is well-defined: notice that $C \cdot A$ is defined since C has domain U and A has codomain U. Furthermore, $C \cdot A$ has domain U and codomain V.

β is injective: if $C \cdot A = C' \cdot A$, then multiplying on the right by A^{-1} gives $C \cdot (A \cdot A^{-1}) = C' \cdot (A \cdot A^{-1})$, i.e., $C \cdot U = C' \cdot U$, and since C and C' are already cosets of U this gives $C = C'$.

β is surjective: given any $B \in \mathcal{W}(G)$ with domain U and codomain V, the product $B \cdot A^{-1}$ is defined since the codomain of B is V, which equals the codomain of A, and hence the domain of A^{-1}. The coset $B \cdot A^{-1}$ has domain U (the domain of B) and codomain U (the domain of A), so it is a two-sided coset of U. Furthermore $(B \cdot A^{-1}) \cdot A = B \cdot (A^{-1} \cdot A) = B \cdot V = B$, so B is in the range of β. □

$\mathrm{Inn}(G)$ denotes the group of inner automorphisms of G. The following implies Theorem 1.1(a). As a hypothesis it would suffice to assume that each open subgroup has finite index in its normaliser.

Theorem 2.7. $\mathrm{Inn}(G)$ *is closed in* $\mathrm{Aut}(G)$ *for each Roelcke precompact group* G.

Proof. Let $(A_n)_{n\in\omega}$ be a listing of the open cosets of G without repetition. Given $\Phi \in \mathrm{Aut}(G)$, we will define a tree T_Φ on the alphabet consisting of pairs of open cosets of G. For this we define T^n_Φ, the n-th level of the tree for every $n < \omega$. The n-level consists of all the pairs of approximations to some

element g and its inverse such that $gA_ig^{-1} = \Phi(A_i)$ for $i < n$. Formally, we let

$$T^n_\Phi = \left\{ \langle (B_i, C_i) \rangle_{i<n} \colon \Phi(A_i) = B_i \cdot A_i \cdot C_i^{-1} \wedge \bigcap_{i<n} (B_i \cap C_i) \neq \emptyset \right\}.$$

Note that the domains of open cosets B_i and C_i as in this definition are determined by the condition that $\Phi(A_i) = B_i \cdot A_i \cdot C_i^{-1}$. Also note that $|N_G(U) : U|$ is finite for each open subgroup U because each two-sided coset of U is a double coset of U. This is implied by our hypothesis that G is Roelcke precompact (Definition 2.1); as mentioned, in fact the presumably weaker condition is sufficient. Fact 2.6 now shows that T_Φ is finitely branching.

Claim. $\Phi \in \mathrm{Aut}(G)$ *is inner if, and only if,* T_Φ *has an infinite path.*

Assuming the claim, we conclude the argument as follows. By König's lemma, T_Φ has an infinite path iff each of its levels T^n_Φ is nonempty. Whether T^n_Φ is nonempty only depends on the values $\Phi(A_0), \ldots, \Phi(A_{n-1})$, so the set of such Φ is clopen in $\mathrm{Aut}(G)$. Thus the condition on Φ that each level of T_Φ is nonempty is closed.

It remains to verify the claim. Suppose A_i is a right coset of a subgroup U_i, and a left coset of a subgroup V_i.

For the implication from *left to right*, suppose that there is $g \in G$ such that $\Phi(A) = gAg^{-1}$ for each open coset A. Now let $B_i = gU_i$ and $C_i = gV_i$. Then $(\langle B_i, C_i \rangle)_{i\in\omega}$ is an infinite path on T_Φ: we have $g \in \bigcap_{i<n}(B_i \cap C_i)$ for each n, and $B_i \cdot A_i \cdot C_i^{-1} = gU_iA_iV_ig^{-1} = gA_ig^{-1} = \Phi(A_i)$.

For the implication from *right to left*, let f be an infinite path on T_Φ, and write $\langle B_i, C_i \rangle = f(i)$. For each n, choose $g_n \in \bigcap_{i<n}(B_i \cap C_i)$. We will show that $\lim_n g_n$ and $\lim_n (g_n^{-1})$ exist. Given m, let $r = r(m)$ be the number such that

$$A_{r(m)} = G_{(0,\ldots,m-1)},$$

where $G_{(0,\ldots,m-1)}$ denotes the pointwise stabilizer of $\{0, \ldots, m-1\}$ in G. Note that B_r and C_r are left cosets of $G_{(0,\ldots,m-1)}$ since $B_r \cdot A_r$ and $A_r \cdot C_r^{-1}$ are defined. If $n \geqslant r$ then for each $i < m$, we have $g_n(i) = g_r(i)$, because $g_n^{-1}g_r \in B_r^{-1} \cdot B_r = G_{(0,\ldots,m-1)}$. Similarly, $g_n^{-1}(i) = g_r^{-1}(i)$ because $g_rg_n^{-1} \in C_r^{-1} \cdot C_r = G_{(0,\ldots,m-1)}$. Thus $g(i) := \lim_n g_n(i)$ and $h(i) := \lim_n g_n^{-1}(i)$ exist for each i, as required.

Clearly the functions $g \colon \omega \to \omega$ and $h \colon \omega \to \omega$ are inverses of each other. Thus $g \in G$, as g is a permutation that is a pointwise limit of elements of G.

For each $i \in \omega$ we have $g = \lim_{n>i} g_n \in B_i \cap C_i$ because $B_i \cap C_i$ is closed. Thus $B_i = gU_i$ and $C_i = gV_i$, and we have for each i

$$gA_ig^{-1} = g(U_i \cdot A_i \cdot V_i)g^{-1} = B_i \cdot A_i \cdot C_i^{-1} = \Phi(A_i),$$

so Φ is inner. □

Remark 2.8. The topology on $\mathrm{Inn}(G)$ inherited from $\mathrm{Aut}(G)$ is the expected one. Indeed, the centre $Z = Z(G)$ of G is closed. We have a Polish topology on G/Z by declaring UZ/Z open iff UZ is open in G. The canonical isomorphism $L\colon G/Z \to \mathrm{Inn}(G)$ is continuous. Since $\mathrm{Inn}(G)$ is Polish as a closed subgroup of $\mathrm{Aut}(G)$, L is a homeomorphism by a standard result in the theory of Polish groups (see [12, 2.3.4]).

We note that a hypothesis such as Roelcke precompactness is needed in 2.7. Wu [32] gave an example of a discrete group L such that $\mathrm{Inn}(L)$ is not closed in $\mathrm{Aut}(L)$. A slight modification yields a nilpotent-2 group of exponent 3. (This group resembles the groups recently studied in [24]; in particular, it is finite automaton presentable.)

Example 2.9 (Similar to Example 4.5 in [32])**.** Let G be the group generated by elements a_i, b_i, c of order 3 ($i \in \mathbb{N}$), where c is central, with the relations, for $i \neq k$

$$b_i a_i b_i^{-1} = a_i c, \quad [a_i, a_k] = [a_i, b_k] = [b_i, b_k] = 1.$$

Let Φ be the automorphism of G given by $\Phi(a_i) = a_i c$, $\Phi(b_i) = b_i$, and $\Phi(c) = c$. It is not inner, but in the closure of $\mathrm{Inn}(G)$.

Proof. To check that Φ is indeed in $\mathrm{Aut}(G)$, note that c can be omitted from the list of generators. Given a word w in a_i, b_i where each letter occurs with exponent 1 or 2, one uses that $\Phi(w) = wc^{m \mod 3}$, where m is the number of occurrences of a_i's. The inverse of Φ is given by $a_i \mapsto a_i c^{-1}$ and as before $b_i \mapsto b_i$, $c \mapsto c$.

Write $g_n = \prod_{i<n} b_i$. We have $g_n a_i g_n^{-1} = a_i c$ for each $i < n$, and $g_n b_i g_n^{-1} = b_i$ for each i. Letting Φ_n be conjugation by g_n, we have $\lim_n \Phi_n = \Phi$ in $\mathrm{Aut}(G)$. For each $g \in G$, conjugation by g fixes almost all the generators. So Φ is not inner. □

3. Some relevant model-theoretic concepts with applications

The language we use in this paper is mainly from the areas of topological groups and permutation groups. The following well-known fact can be used as a mechanism to translate from group theoretic language to model theoretic language and back. The notation M_G below should not be confused with the notation of a coarse group $\mathcal{M}(G)$ from [23].

Fact 3.1 (see [13], Theorem 4.1.4)**.** Let G be oligomorphic. We have $G = \mathrm{Aut}(M_G)$ for the $\aleph_0$-categorical relational structure M_G that has an n-ary relation P_i^n for each n-orbit of G. In this context one says that M_G is a *canonical structure* for G; it is unique up to permuting the names of the k-orbits, for each k.

The operation $G \mapsto M_G$ is easily seen to be Borel [23, Section 1.3]. Recall that for oligomorphic groups G, H, we have that $G \cong H$ iff M_G and M_H are bi-interpretable (for a definition of bi-interpretability see [13, Chapter 5]). The model theoretical language is of great importance in proving results in the setting of oligomorphic groups. For instance, it is not hard to see that bi-interpretability of $\aleph_0$-categorical structures is Borel [23, Section 1.3]; so topological isomorphism of oligomorphic groups is also Borel (where naively it looks merely analytical). Furthermore, using that the operation $G \mapsto M_G$ is Borel, one can verify that classes of groups such as the ones having no algebraicity (Definition 5.1) are Borel by looking at the corresponding classes of structures.

This section uses model theoretic language to provide results that will be important later on. For instance, we get a grip on the open cosets of G by representing such a coset as a pair of imaginaries of the same sort.

3.1. Imaginaries and M^{eq}

The structure M^{eq} of imaginaries over a structure M goes back to Shelah [28]. We follow the treatment in Hodges [13, p. 151] (see also [7, pg. 12–13]). Recall that 'definable' means $\emptyset$-definable by Convention 1.3. The structure M^{eq} has sorts $S = D/E$, where $D \subseteq M^k$ (for $1 \leqslant k < \omega$) is definable and E is a definable equivalence relation on D; its elements are written as $\bar{a}/E$ for $\bar{a} \in D$. The atomic relations of M^{eq} are the atomic relations of the given structure M (viewed as relations on the "home sort" M), as well as the graphs of the projections $D \to S = D/E$ for each sort S.

For sorts $S_i = D_i/E_i$, $i = 1, \ldots, n$, we have the product sort $\prod S_i = \prod D_i / \prod E_i$, where the equivalence relation $\prod E_i$ is defined component-wise.

In a many-sorted structure, each definable relation is a subset of $\prod_{i<n} S_i$ for sorts $S_0, \ldots, S_{n-1}$. The definable relations are given as usual via formulas in the first-order logic in the corresponding signature with variables for each sort. Extending the usual definition of algebraic/definable closure to a many-sorted structure is no problem. We note that when M is $\aleph_0$-categorical, $\mathrm{acl}(A) \cap S$ is finite for each finite $A \subseteq M^{\mathrm{eq}}$ and each sort S.

3.2. Orbital structures and profinite groups

A *reduct* of a structure M is a structure N with the same domain such that each relation or function of N is definable in M. (In some references the notion of "reduct" is limited to restrictions of the language; the two notions can be reconciled using Morleyzation in the sense of [13, pg. 63].)

The reduct relation is a quasi-order among structures with a given domain; the corresponding equivalence relation is interdefinabilty, namely, that two structures have the same definable relations. It is well known that for $\aleph_0$-categorical structures N, M, N is a reduct if M iff $\mathrm{Aut}(M)$ is a subgroup of $\mathrm{Aut}(N)$.

Definition 3.2 (Orbital structure)**.** Consider the language $\{\sim_n : 0 < n < \omega\}$ where for each n, the relation symbol $\sim_n$ is of arity $2n$. Given an $\aleph_0$-categorical structure M with domain ω, by $\mathcal{E}_M$ we denote the structure in this language such that $\bar{a} \sim_n \bar{b}$ iff there is $g \in \mathrm{Aut}(M)$ such that $g(\bar{a}) = \bar{b}$. One says that $\mathcal{E}_M$ is the *orbital structure* of M.

It is easy to check that the structure $\mathcal{E}_M$ from Definition 3.2 is a reduct of M.

By $\mathcal{N}_G$ we denote the normaliser of G in $\mathrm{Sym}(\omega)$. By the following, $\mathrm{Aut}(\mathcal{E}_M)$ is the normaliser of $G = \mathrm{Aut}(M)$ in $\mathrm{Sym}(M)$, the group of permutations of M.

Fact 3.3. Let M be an $\aleph_0$-categorical structure with domain ω. Let $G = \mathrm{Aut}(M)$.

(i) $\mathcal{N}_G = \mathrm{Aut}(\mathcal{E}_M)$.
(ii) $\mathcal{E}_M$ is up to interdefinability the smallest reduct L of M such that $\mathrm{Aut}(M)$ is a normal subgroup of $\mathrm{Aut}(L)$.

Proof. (i) To show that $\mathcal{N}_G \leq \mathrm{Aut}(\mathcal{E}_M)$, let $h \in \mathcal{N}_G$. Suppose that $g \cdot \bar{a} = \bar{b}$ where $g \in G$ and $\bar{a}, \bar{b} \in M^n$. Let $g' = g^h \in G$. Then $g'h \cdot \bar{a} = h \cdot \bar{b}$. So h preserves the relations $\sim_n$.

For the converse inclusion, we may assume that $M = M_G$. If $h \in \mathrm{Aut}(\mathcal{E}_M)$ and g preserves each relation P_i^n, then so does g^h. Thus $h \in \mathcal{N}_G$.
(ii) Let R be a reduct of M. By definition of a normaliser, G is normal in $\mathrm{Aut}(R)$ iff $\mathrm{Aut}(R) \leq \mathcal{N}_G$. By (i) this is equivalent to $\mathrm{Aut}(R)$ being contained in $\mathrm{Aut}(\mathcal{E}_M)$, which in turn is equivalent to $\mathcal{E}_M$ being a reduct of R. □

Theorem 3.4. *Let M be an $\aleph_0$-categorical structure. Then $V = \mathrm{Aut}(\mathcal{E}_M)/\mathrm{Aut}(M)$ is a profinite group. If M is interdefinable with a structure in a finite signature, then V is finite.*

Proof. Since M is interdefinable with $M_{\mathrm{Aut}(M)}$, we may assume that for every n with $0 < n < \omega$, the structure M has predicates $P^n_1, \dots, P^n_{k(n)}$ for each of its n-orbits, and no other relations or functions. For $n \leqslant \omega$ let M_n be the structure obtained after removing the predicates of arity greater than n, and let $\mathcal{E}_n := \mathcal{E}_{M_n}$ be the corresponding orbital structure as in Definition 3.2.

For $k \leqslant m \leqslant \omega$ the identity map $\mathcal{E}_m \to \mathcal{E}_k$ induces a map

$$q_{m,k} \colon \mathrm{Aut}(\mathcal{E}_m)/\mathrm{Aut}(M_m) \to \mathrm{Aut}(\mathcal{E}_k)/\mathrm{Aut}(M_k).$$

Writing $p_n = q_{n+1,n}$, this yields an inverse system

$$(\mathrm{Aut}(\mathcal{E}_n)/\mathrm{Aut}(M_n), p_n)_{n\in\mathbb{N}}. \qquad (*)$$

The group $\mathrm{Aut}(\mathcal{E}_n)/\mathrm{Aut}(M_n)$ is canonically isomorphic to a subgroup of $S_{k(n)}$ (the group of permutations of $\{1, \dots, k(n)\}$): an isomorphism is induced by mapping $\pi \in \mathrm{Aut}(\mathcal{E}_n)$ to the permutation $\alpha \in S_{k(n)}$ such that $\pi(P^n_i) = P^n_{\alpha(i)}$ for each $i \leqslant k(n)$. Let

$$L := \varprojlim_n (\mathrm{Aut}(\mathcal{E}_n)/\mathrm{Aut}(M_n), p_n)$$

be the inverse limit in the category of topological groups, where the finite groups $\mathrm{Aut}(\mathcal{E}_n)/\mathrm{Aut}(M_n)$ carry the discrete topology. Every $f \in L$ can be concretely seen as a certain sequence of permutations $(\alpha_n)_{0<n<\omega}$ where $\alpha_n \in S_{k(n)}$. *We claim that $V \cong L$ via the continuous homomorphism $F \colon V \to L$ induced by the maps $q_{\omega,n} : V \to \mathrm{Aut}(\mathcal{E}_n)/\mathrm{Aut}(M_n)$.* To verify this, we invoke the universal property of the inverse limit. To see that F is 1-1, suppose $\pi \in \mathrm{Aut}(\mathcal{E}_\omega) \setminus \mathrm{Aut}(M)$. Then there is n and $i \leqslant k(n)$ such that $\pi(P^n_i) = P^n_j$ with $i \neq j$. So the image of $\pi\mathrm{Aut}(M)$ in $\mathrm{Aut}(\mathcal{E}_n)/\mathrm{Aut}(M_n)$ is not equal to the identity. Thus $F(\pi\mathrm{Aut}(M))$ is not equal to the identity.

To see that F is onto, given $f \in L$, let $(\alpha_n)_{0<n<\omega}$ be as above, and let T_f be the theory in the language of M extended by an additional function symbol π, and with the following set of axioms:

(1) $\mathrm{Th}(M)$;
(2) "π is a permutation";
(3) "$\pi(P^n_i) = P^n_{\alpha_n(i)}$, for each n and each $i \leqslant k(n)$".

Note that T_f is consistent by the Compactness Theorem, using the assumption that each α_n is induced by an automorphism π_n of $\mathcal{E}_n$. Let N be any countable model of T_f, and let N' be the reduct of N to the language of M. Since M is $\aleph_0$-categorical, there is an isomorphism $h \colon M \to N'$. We have $g =$

$h^{-1} \circ \pi^N \circ h \in \mathrm{Aut}(\mathcal{E}_M)$ and $F(g\mathrm{Aut}(M)) = f$. As $f \in L$ was arbitrary this shows that F is onto.

The group isomorphism $F^{-1}\colon L \to V$ is clearly Borel, so it is continuous as a Borel (and hence Baire measurable) homomorphism between Polish groups; see, e.g., [12, Th. 2.3.3].

Suppose the structure M is interdefinable with a structure in a finite signature, and let n be the maximum arity of a symbol in the signature. Then for each $r \geq n$, the structure M_r is an extension by first-order definitions of M_n, and hence $\mathrm{Aut}(M_r) = \mathrm{Aut}(M_n)$, and $\mathrm{Aut}(\mathcal{E}_r) = \mathrm{Aut}(\mathcal{E}_n)$. Hence the inverse system $(*)$ above is eventually constant, whence its inverse limit is finite. □

Corollary 3.5. *Let M be $\aleph_0$-categorical. Let L be a reduct of M such that $\mathrm{Aut}(M)$ is a normal subgroup of $\mathrm{Aut}(L)$. Then $\mathrm{Aut}(L)/\mathrm{Aut}(M)$ is a profinite group.*

Proof. $\mathrm{Aut}(L)$ is contained $\mathrm{Aut}(\mathcal{E}_M)$ by Fact 3.3. Since $\mathrm{Aut}(L)$ is also closed in $\mathrm{Sym}(\omega)$, we conclude that $\mathrm{Aut}(L)/\mathrm{Aut}(M)$ is profinite as a closed subgroup of $\mathrm{Aut}(\mathcal{E}_M)/\mathrm{Aut}(M)$. □

3.3. Open cosets and pairs of imaginaries

In this section, G will be an oligomorphic closed subgroup of $\mathrm{Sym}(\omega)$. Recall that $G = \mathrm{Aut}(M)$ where $M = M_G$ is the canonical structure for G. Note that G acts naturally on M^{eq}. Here we describe open cosets and the inclusion relation between them model-theoretically via M^{eq}. We begin with open subgroups. The following technical notion will be needed in Section 5.1.

Definition 3.6 ([27])**.** Let M be a set, $D \subseteq M^n$ for $n \geqslant 1$, and let E be an equivalence relation on D. The relation E is called *degenerate* if for some set $\sigma \subsetneq n$, the equivalence relation $E_{\sigma,D} = \{\langle a, b\rangle \in D^2 \colon \forall i \in \sigma\, [a_i = b_i]\}$ is contained in E. Otherwise E is called *non-degenerate*.

For $\alpha \in M^{\mathrm{eq}}$, let $G_\alpha = \{g \in G \colon g(\alpha) = \alpha\}$. The open subgroups coincide with the groups of the form G_α by the following well-known fact that is implicit in the proof of [1, Theorem 1.2]. The present form is close to [7, Theorem 2.7].

Lemma 3.7. *Let H be an open subgroup of G. Suppose that $G_{\bar{a}} \leqslant H$ where $\bar{a} \in M^n$ (note that such an $\bar{a}$ exists as H is open).*

(i) There are definable relations $D \subseteq M^n$ and $E \subseteq M^{2n}$ such that E induces an equivalence relation on D, and $H = G_{\bar{a}/E}$ for some $\bar{a} \in D$.

(ii) *For every $\bar{a} \in M^n$, only finitely many open subgroups H of G contain $G_{\bar{a}}$.*

(iii) *If a tuple $\bar{a}$ is chosen of minimal length such that $G_{\bar{a}} \leqslant H$, then E, seen as an equivalence relation on D, is non-degenerate.*

Proof. (i) Let $D = G \cdot \bar{a}$, and let the equivalence relation E on D be given by

$$h\bar{a}Eg\bar{a} \leftrightarrow g^{-1}h \in H.$$

Since D, as well as E viewed as a $2n$-ary relation, are invariant under the action of G, they are both definable. Let $\alpha = \bar{a}/E$. Then we have $G_\alpha = H$, because $g(\alpha) = \alpha$ iff $g(\bar{a})E\bar{a}$ iff $g \in H$.
(ii) is immediate from (i) because M has only finitely many $2n$-orbits.
(iii) Assume for a contradiction that E is degenerate via σ. We may assume that $\sigma = \{0, \dots, n-2\}$. Let $\bar{a}'$ be $\bar{a}$ with the last entry removed. We claim that $G_{\bar{a}'} \leqslant H$. For let $h \in G_{\bar{a}'}$, and let $\bar{b}$ be the tuple $\bar{a}'$ with $h(a_{n-1})$ appended, so that $\bar{b} = h(\bar{a})$. Then $\bar{b}\, E_{\sigma,D}\, \bar{a}$ and hence $\bar{b}\, E\, \bar{a}$ hold, so $h \in H$ by the definition of E. □

The containment relation between open subgroups is described by relative definability in M^{eq}.

Fact 3.8. For $\alpha, \beta \in M^{\mathrm{eq}}$, we have $G_\alpha \subseteq G_\beta \Leftrightarrow \beta \in \mathrm{dcl}(\alpha)$.

Proof. The implication "$\Leftarrow$" is immediate.

For the implication "$\Rightarrow$", suppose $\alpha = \bar{a}/E$ and $\beta = \widetilde{b}/F$ as above. Then

$$h\bar{a}Eg\bar{a} \leftrightarrow g^{-1}h \in G_\alpha \text{ and } h\widetilde{b}Fg\widetilde{b} \leftrightarrow g^{-1}h \in G_\beta.$$

Let $R = G \cdot \langle \bar{a}, \widetilde{b} \rangle$, and note that R is a left invariant, and hence definable, relation. Since $G_\alpha \subseteq G_\beta$, β is the unique equivalence class of F containing $\widetilde{b}'$ for some $\langle \bar{a}', \widetilde{b}' \rangle \in R$ with $\bar{a}' \in \alpha$. Hence $\beta \in \mathrm{dcl}(\alpha)$. □

Next we represent open cosets by pairs of imaginaries in the same orbit of M^{eq}.

Notation 3.9. For imaginaries α_0, α_1 of the same sort of M^{eq}, let

$$[\alpha_0, \alpha_1] = \{g \in G \colon g(\alpha_1) = \alpha_0\}.$$

If $[\alpha_0, \alpha_1] \neq \emptyset$, then clearly $[\alpha_0, \alpha_1]$ is a right coset of G_{α_0}, and a left coset of G_{α_1}. Also $[\alpha_1, \alpha_0] = [\alpha_0, \alpha_1]^{-1}$. We note that for each sort S, the relation $\{\langle \alpha_0, \alpha_1 \rangle \in S^2 \colon [\alpha_0, \alpha_1] \neq \emptyset\}$ is definable: it says that α_0 and α_1 are in the same orbit under the action of G on M^{eq}; now use that each orbit on M^{eq} is definable.

Fact 3.10. Each open coset A is of the form $[\alpha_0, \alpha_1]$. Here α_1 can be chosen to be any element of M^{eq} such that A is a left coset of G_{α_1} (which exists by 3.7(i)).

Proof. Note that $A = hG_{\alpha_1}$ for any $h \in A$; now let $\alpha_0 = h(\alpha_1)$. □

Inclusion of open cosets, and the product of cosets in case it is defined in $\mathcal{W}(G)$, can also be expressed in terms of definability; see [20, 4.8].

Example 3.11. Let $k \geqslant 2$. Let M be the countably infinite structure with one permutation π, where each cycle has length k. Then $G := \mathrm{Aut}(M) \cong C_k \wr \mathrm{Sym}(\omega)$ where C_k is a cyclic group of size k, and the wreath product is taken via the action of $\mathrm{Sym}(\omega)$ on ω. Note that G acts 1-transitively on M. Let $V = G_b$ where $b \in M$. Each subgroup U of the form G_a has k right cosets A_i $(i < k)$ that are left cosets of V: these are the $A_i = \{f \in G \colon \pi^i(a) = f(b)\}$ where $0 \leqslant i < k$. Letting $a = b$ we see that the group $N_G(V)/V$ is cyclic of size k.

In general, if $G = \mathrm{Aut}(M)$ is 1-transitive and $V = G_b$ for some $b \in M$, then $N_G(V)/V$ is isomorphic to the centraliser of G in $\mathrm{Sym}(\omega)$ via the map $\pi \to [\pi(b), b]$. The centraliser is non-abelian in a variant of the above example: replace the cyclic group above by a subgroup of S_n with non-abelian centraliser.

3.4. Out(G) is t.d.l.c. for oligomorphic groups G

Recall that $\mathrm{Inn}(G)$ is the group of inner automorphisms of G, which by 2.7 is a closed subgroup of $\mathrm{Aut}(G)$ whenever G is Roelcke precompact. We now obtain Theorem 1.1(c).

Theorem 3.12. *Let G be an oligomorphic group, and $\mathcal{N}_G$ its normaliser in* $\mathrm{Sym}(\omega)$.

(i) Let $S \colon \mathcal{N}_G \to \mathrm{Aut}(G)$ be the homomorphism defined by $S(\pi)(g) = \pi \circ g \circ \pi^{-1}$. The subgroup $\mathrm{range}(S)$ *is open in* $\mathrm{Aut}(G)$.

(ii) $\mathrm{Out}(G)$ *is totally disconnected and locally compact, having the profinite group $\mathcal{P} = \mathrm{range}(S)/\mathrm{Inn}(G)$ as an open subgroup. If M_G is interdefinable with a structure in a finite signature, then $\mathcal{P}$ is finite, and hence* $\mathrm{Out}(G)$ *is discrete.*

Proof. (i). We will use Notation 3.9, where M is the canonical structure for G as in 3.1 (with domain ω). Note that for $a, c, d \in \omega$, if $[d, a] \supseteq [c, a] \neq \emptyset$ then $d = c$.

Let $a_1, \dots, a_n$ represent the 1-orbits of G. It suffices to show that the open subgroup $\{\Phi \in \mathrm{Aut}(G) \colon \Phi(G_{a_i}) = G_{a_i} \text{ for each } i\}$ is contained in range(S). Suppose that Φ is in this subgroup. Since Φ fixes each subgroup G_{a_i}, for each coset $D = [b, a_i]$, $\Phi(D)$ is also a left coset of G_{a_i}. Hence, by 3.10, $\Phi(D)$ can be written in the form $[d, a_i]$, and d is unique as noted above. Define a function $\pi_\Phi \colon \omega \to \omega$ by

$$\pi_\Phi(b) = d \text{ if } \Phi([b, a_i]) = [d, a_i], \text{where } [b, a_i] \neq \emptyset$$

(noting that i is uniquely determined by b). Clearly π_Φ and $\pi_{\Phi^{-1}}$ are inverses. So $\pi_\Phi \in \mathrm{Sym}(\omega)$. The following establishes (i).

Claim. *$\pi_\Phi \in \mathcal{N}_G$ and $R(\pi_\Phi) = \Phi$.*

Write $\pi = \pi_\Phi$. To verify the claim, we first show that $\Phi([r, s]) = [\pi(r), \pi(s)]$ for each $r, s \in \omega$ such that $[r, s] \neq \emptyset$. Note that by hypothesis on Φ and since $G_{a_i} = [a_i, a_i]$, we have $\pi(a_i) = a_i$ for each i. Also note that $\Phi([b, c])^{-1} = \Phi([c, b])$ for each $b, c \in \omega$. Let now i be such that r and s are in the 1-orbit of a_i. We have $[r, s] = [r, a_i] \cdot [a_i, s] = [r, a_i] \cdot [s, a_i]^{-1}$. So

$$\Phi([r, s]) = \Phi([r, a_i]) \cdot \Phi([s, a_i])^{-1} = [\pi(r), a_i] \cdot [\pi(s), a_i]^{-1} = [\pi(r), \pi(s)].$$

Next, for each $h \in G$ we have $\{h\} = \bigcap_{s \in \omega} hG_s$, and $hG_s = [h(s), s]$ using Notation 3.9. Then

$$\begin{aligned} \{\Phi(g)\} &= \bigcap_{s \in \omega} \Phi([g(s), s]) = \bigcap_{s \in \omega} [\pi(g(s)), \pi(s)] \\ &= \bigcap_{t \in \omega} [\pi(g(\pi^{-1}(t))), t] = \{g^\pi\}, \end{aligned}$$

as required. This shows the claim.

(ii). The kernel of the homomorphism S is the centraliser $C = C_{\mathrm{Sym}(\omega)}(G)$ of G in $\mathrm{Sym}(\omega)$, which consists of the permutations that are definable in M when seen as binary relations on M. Note that C is finite because M is ω-categorical. By general group theory C is a normal subgroup of $\mathcal{N}_G$, hence GC is also a normal subgroup of $\mathcal{N}_G$. So G has finite index in GC, whence GC is closed in $\mathrm{Sym}(\omega)$. Since C is the kernel of R and $S(G) = \mathrm{Inn}(G)$, we have $S^{-1}(\mathrm{Inn}(G)) = GC$. Thus range$(S)/\mathrm{Inn}(G)$ is topologically isomorphic to $\mathcal{N}_G/GC$. By Fact 3.3 $\mathcal{N}_G = \mathrm{Aut}(\mathcal{E}_M)$, so by the first statement of Theorem 3.4 the group $\mathcal{N}_G/G$ is compact. So $\mathcal{N}_G/GC$ is also compact as its topological quotient. If M_G is interdefinable with a structure in a finite signature, then $\mathcal{P}$ is finite by the second statement of Theorem 3.4. □

Remark 3.13. Using the model theoretic setting of bi-interpretations, Schlicht and the first author [22, Section 3] obtain results related to 3.12. The group $\mathrm{Out}(G)$ corresponds to the group of self bi-interpretations of $\mathrm{Th}(M)$ up to an appropriate congruence relation, and $\mathrm{range}(R)/\mathrm{Inn}(G)$ corresponds to the group of one-dimensional such bi-interpretations up to this congruence. If M_G is interdefinable with a structure in a finite signature, then this group is clearly finite. This gives a perhaps more instructive proof that $\mathrm{Out}(G)$ is discrete in this case.

4. A general result involving a Borel property of open subgroups

Recall that by convention, isomorphisms between topological groups are assumed to be topological. The following set-up resembles the one in [17, Th. 3.1]; we refer to that source for background. In particular, the set of closed subgroups of $\mathrm{Sym}(\omega)$ forms a standard Borel space denoted $\mathcal{U}(S_\infty)$, and its subset of oligomorphic groups is Borel.

Definition 4.1. Let $\mathcal{V}$ be a Borel class of oligomorphic groups. Let P be a Borel relation picking open subgroups from groups $G \in \mathcal{V}$. If $P(U, G)$ holds, we say that U is a *P-subgroup of G*.

(a) P is called *invariant* if any isomorphism between groups $G, H \in \mathcal{V}$ preserves P.
(b) An invariant Borel relation P is called *sub-basic for $\mathcal{V}$* if each open subgroup of a group $G \in \mathcal{V}$ contains a finite intersection of P-subgroups of G.

Assumption 4.2. In the following we fix a Borel class $\mathcal{V}$ of oligomorphic groups, and assume that P is an invariant Borel relation that is sub-basic for $\mathcal{V}$.

For each $G \in \mathcal{V}$ we define a countable structure $\mathcal{C}_G^P$ that is a "trimmed-down" version of the meet groupoid $\mathcal{W}(G)$ of Definition 2.5. The domain of $\mathcal{C}_G^P$ consists of the cosets of P-subgroups, and in place of the meet operation it has merely non-disjointness relations. A structure of form $\mathcal{C}_G^P$ is similar to the structures in [26, Definition 2.15] used in the particular case of groups of the form $\mathrm{Aut}(M)$ where M has weak elimination of imaginaries. However, to get the duality between groups in $\mathcal{V}$ and their corresponding structures expressed in Lemma 4.4 below, the partial product operation is needed; this operation was absent from [26, Definition 2.15].

Definition 4.3. Let $G \in \mathcal{V}$. We form a structure $\mathcal{C}_G^P$ on the set

$$\{gU : U \text{ is a } P\text{-subgroup of } G \text{ and } g \in G\}$$

with the following.

(a) Unary functions Dom and Cod (which stand for "domain" and "codomain") such that $\mathrm{Dom}(B) = U$ iff B is a right coset of the subgroup U, and $\mathrm{Cod}(B) = V$ iff B is a left coset of the subgroup V.
(b) For each n, the n-ary relation I_n holds of $(B_1, \dots, B_n)$ if $B_1 \cap \dots \cap B_n \neq \emptyset$.
(c) A ternary relation holds of (A, B, C) if $\mathrm{Cod}(A) = \mathrm{Dom}(B)$ and $AB = C$. As before in the case of $\mathcal{W}(G)$, we write $A \cdot B = C$ for this.

We will omit the superscript P in $\mathcal{C}_G^P$ if P is understood from the context. The next lemma establishes the duality between groups G in $\mathcal{V}$ and the corresponding structures $\mathcal{C}_G$.

Lemma 4.4. *Let $G, H \in \mathcal{V}$. If α is an isomorphism $G \to H$, let $R(\alpha)\colon \mathcal{C}_G \to \mathcal{C}_H$ be the map given by $B \mapsto \alpha(B)$.*

(1) $R(\alpha)$ is an isomorphism of the structures $\mathcal{C}_G$ and $\mathcal{C}_H$.
(2) Each isomorphism $\theta : \mathcal{C}_G \to \mathcal{C}_H$ is of the form $R(\alpha)$ for a unique α.
(3) The map R restricted to $\mathrm{Aut}(G)$ is an isomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathcal{C}_G)$.

Proof. We have $G = \mathrm{Aut}(M)$ and $H = \mathrm{Aut}(N)$ where M, N are the canonical structures for G and H, respectively.
(1) Given that α is a topological isomorphism, it only remains to check that the relations Dom and Cod are preserved in both directions by $R(\alpha)$. For all elements B, U, V in a structure of the form $\mathcal{C}_L$, we have $\mathrm{Dom}(B, U)$ iff $B \cdot B^{-1} = U$, and $\mathrm{Cod}(B, V)$ iff $B^{-1} \cdot B = V$. This suffices because by its definition, $R(\alpha)$ preserves inversion.
(2) Let $(U_i)_{i<\omega}$ list the subgroups in $\mathcal{C}_G$, and let $V_i := \theta(U_i)$. Then $(V_i)_{i<\omega}$ lists the subgroups in $\mathcal{C}_H$. We define an isomorphism $\alpha\colon G \to H$ such that $R(\alpha) = \theta$. Given $g \in G$, first we define the value $\alpha(g)$, which will be a function $\omega \to \omega$. Let $A_i = gU_i$ and $B_i = \theta(A_i)$. For each n we have $\mathcal{C}_G \models I_n(A_1, \dots, A_n)$. So, since θ is an isomorphism, there is $r_n \in \bigcap_{i<n} B_i$. Since $\mathrm{Cod}(B_i, V_i)$ holds in $\mathcal{C}_H$, we have $r_n V_i = B_i$ for each $i < n$. Since P is sub-basic for H, for each $b \in N$ there is a finite set $I \subset \omega$ such that $\bigcap_{i \in I} V_i \subseteq H_b$. So

$$\alpha(g)(b) \colon = (\lim_n r_n)(b)$$

exists for each b.

We verify that α is a homomorphism $G \to H$. For the rest of this proof, to avoid an awkward buildup of parentheses due to iterated function applications, we use the following notation: instead of $f(x)$ we write $f.x$ (read as "f of x"). So $f(x)(z)$ becomes $f.x.z$.

First we verify that $\alpha.e_G = e_H$. For $g = e_G$, in the definition of $\alpha.g$ we have $A_i = U_i$ and so $B_i = V_i$; thus $r_n \in \bigcap_{i<n} V_i$. For any $x \in \omega$, we have $\alpha.e_G.x = \lim_n r_n.x$. Since $r_n \in \bigcap_{i<n} V_i$, this implies $\alpha.e_G.x = x$.

Next we verify that for each $g, h \in G$, we have $(\alpha.h) \circ (\alpha.g) = \alpha.(hg)$. Let $U_i' = gU_ig^{-1}$ and $V_i' = \theta(U_i')$. Also let $A_i = \theta(gU_i)$ and $B_i = \theta(hU_i')$. Given $x \in \omega$ write $y = \alpha.g.x$. Let n be so large that $\bigcap_{i<n} V_i \subseteq H_x \cap H_y$. Fix $r \in \bigcap_{i<n} A_i$. Then $r.x = y$. Since P is invariant as defined in 4.1, the class of P-subgroups of G is closed under conjugation. So let $m \geqslant n$ be so large that each U_i, $i < n$, is of the form U_j' for some $j < m$. Let $s \in \bigcap_{i<m} B_i$. Then $s \in \bigcap_{i<n} \theta(hU_i)$, and hence $s.y = \alpha.h.y$.

Since $hU_i' \cdot gU_i = hgU_i$ holds in $\mathcal{C}_G$ and $\theta : \mathcal{C}_G \to \mathcal{C}_H$ is an isomorphism, for each i, $B_i \cdot A_i = \theta(hgU_i)$ holds in $\mathcal{C}_H$. Thus $s \circ r \in \bigcap_{i<n} \theta(hgU_i)$. By the choice of n and since $\mathrm{Cod}(\theta(hgU_i)) = V_i$, this shows that $(\alpha.h) \circ (\alpha.g)(x) = s \circ r(x) = \alpha.(hg).x$.

Letting $h = g^{-1}$, we infer that $\alpha(g) \in \mathrm{Sym}(\omega)$. Then clearly $\alpha(g)$ is in H because H is a closed subset of $\mathrm{Sym}(\omega)$; recall here that $\alpha(g) = \lim_n r_n$ where $r_n \in H$, for all $n < \omega$.

Exchanging the roles of $\mathcal{C}_G$ and $\mathcal{C}_H$ and using $\theta^{-1} \colon \mathcal{C}_H \to \mathcal{C}_G$, we find an inverse of α. Thus $\alpha : G \to H$ is a group isomorphism. To see that α is continuous at e_G (and hence continuous), it suffices to show that $\alpha^{-1}(H_x)$ contains an open subgroup, for each $x \in N$. As before let n be such that $\bigcap_{i<n} V_i \subseteq H_x$; then $\bigcap_{i<n} U_i$ is the required subgroup. Similarly, α^{-1} is continuous.
(3) For each $\alpha, \beta \in \mathrm{Aut}(G)$, we have $R(\alpha\beta^{-1}) = R(\alpha)R(\beta)^{-1}$ by the definition of R, so R is an isomorphism of groups. To see that R is continuous (and hence a homeomorphism using [12, 2.3.4]), note that a basic neighbourhood of the identity in $\mathrm{Aut}(\mathcal{C}_G)$ has the form $\{\theta \colon \bigwedge_{i=1}^n \theta(A_i) = A_i\}$, where $A_1, \ldots, A_n \in \mathcal{C}_G$. Its preimage under R is open in $\mathrm{Aut}(G)$ according to Definition 2.3. □

By Lemma 4.4(3) and the definition of its topology, $\mathrm{Aut}(G)$ acts continuously on $\mathcal{C}_G$ by automorphisms, namely, $\Phi \cdot A = \Phi(A)$ where $\Phi \in \mathrm{Aut}(G)$ and $A \in \mathcal{C}_G$. Under an additional hypothesis, the continuous subaction of the closed subgroup $\mathrm{Inn}(G)$ on $\mathcal{C}_G$ is oligomorphic.

Lemma 4.5. *Let $G \in \mathcal{V}$. Assume that there are only finitely many conjugacy classes of P-subgroups in G. Then the canonical action by automorphisms of* $\mathrm{Inn}(G)$ *on $\mathcal{C}_G$ is oligomorphic.*

Proof. By 2.8, $\mathrm{Inn}(G)$ is topologically isomorphic to $G/Z(G)$, and hence Roelcke precompact. Thus, by [29, Theorem 2.4] it suffices to show that the action of $\mathrm{Inn}(G)$ on $\mathcal{C}_G$ has only finitely many 1-orbits.

Recall that for a pair U, V of conjugate open subgroups, by 2.6 there are $[N_U : U]$ right cosets of U that are left cosets of V, and $[N_U : U]$ is finite as

G is Roelcke precompact. So it suffices to show that there are only finitely many $\mathrm{Inn}(G)$-orbits of pairs (U, V) where U, V are conjugate P-subgroups; the $\mathrm{Inn}(G)$-orbit of $A \in \mathcal{C}_G$ is then given by the orbit of U, V where U is the domain and V the codomain of A, together with the information which right coset of U the element A belongs to.

By hypothesis U, V are of the form G_α and G_β, where α, β are both elements of one of finitely many 1-transitive sorts of M^{eq}. The structure M^{eq} has only finitely many orbits of such pairs α, β. □

Now let $\mathcal{C}_G^+$ be the expansion of $\mathcal{C}_G$ obtained by naming the n-orbits of the action of $\mathrm{Inn}(G)$ on $\mathcal{C}_G$ by predicates, for all $n \geqslant 1$. Note that $\mathcal{C}_G$ is a reduct of $\mathcal{C}_G^+$.

Fact 4.6. Suppose that G is as in Lemma 4.5. Then $\mathcal{C}_G^+$ is $\aleph_0$-categorical, and so $\mathcal{C}_G$ is $\aleph_0$-categorical as well.

Proof. By 4.5 the action by automorphisms of $\mathrm{Inn}(G)$ on $\mathcal{C}_G$ is oligomorphic. By definition, each such automorphism is also an automorphism of $\mathcal{C}_G^+$. Thus $\mathrm{Aut}(\mathcal{C}_G^+)$ is oligomorphic, and hence $\mathcal{C}_G^+$ is $\aleph_0$-categorical. □

Groups $G \leqslant_c \mathrm{Sym}(\omega)$ such that G is (topologically) isomorphic to an oligomorphic group are called *quasi-oligomorphic* in [23, Section 4.3]. The authors show that this property is Borel, and the complexity of their isomorphism relation is Borel equivalent to the isomorphism relation between oligomorphic groups.

Theorem 4.7. *In addition to Assumption 4.2, suppose that for each $G \in \mathcal{V}$, there are only finitely many conjugacy classes of P-subgroups.*

(1) Topological isomorphism on $\mathcal{V}$ is smooth.
(2) $\mathrm{Aut}(G)$ is quasi-oligomorphic for each $G \in \mathcal{V}$.
(3) $\mathrm{Out}(G)$ is profinite for each $G \in \mathcal{V}$.

Proof. (1) The structures with domain ω in a fixed countable signature L can be seen as a standard Borel space $\mathrm{Mod}(L)$ [12, Section 3.5]. By [17, Section 3.3], there is a Borel function taking an oligomorphic group G to a bijection $\nu_G \colon \omega \to \mathrm{dom}(\mathcal{M}(G))$ (this uses a result of Lusin-Novikov in the version of [16, 18.10]). Via this bijection we can view $\mathcal{M}(G)$ as a structure with domain ω in such a way that the map $G \mapsto \mathcal{M}(G)$ is Borel. The structures $\mathcal{M}(G)$ and the meet groupoid $\mathcal{W}(G)$ are interdefinable in a uniform way [23, Remark 2.16], so the map $G \to \mathcal{W}(G)$ is also Borel in this sense. Then, since P is a Borel relation, the map $G \mapsto \mathcal{W}^P(G)$ is Borel, where we define $\mathcal{W}^P(G)$ as the structure $\mathcal{W}(G)$ extended by a unary predicate saying that a subgroup

satisfies P. The domain of the structure $\mathcal{C}_G^P$ can be first-order defined in $\mathcal{W}^P(G)$ as the set of cosets of subgroups that satisfy the property P.

We claim that each relation and operation of $\mathcal{C}_G^P$ in Definition 4.3 can also be defined in $\mathcal{W}^P(G)$ by a first-order formula that does not depend on G (and also not on P). For (a) in 4.3, this holds because $\mathrm{Dom}(B) = U$ iff $B \cdot B^{-1} = U$ and $\mathrm{Cod}(B) = V$ if $B^{-1} \cdot B = V$. For (b) in 4.3, one uses that $\mathcal{W}^P(G)$ contains the binary intersection operation and a constant for $\emptyset$, and for (c) it is immediate.

Viewing $\mathcal{W}^P(G)$ as a structure with domain ω as above, our claim shows that the map $G \mapsto \mathcal{C}_G^P$ is Borel. By Lemma 4.4 and Fact 4.6, the Borel map $G \mapsto \mathrm{Th}(\mathcal{C}_G^P)$ now establishes smoothness of $\cong$ on $\mathcal{V}$.

(2) By Lemma 4.4(3), $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathcal{C}_G^P)$. Now apply Fact 4.6.

(3) Let R be as in Lemma 4.4 for $H = G$. By the definition of $\mathcal{C}_G^+$, the closure of $R(\mathrm{Inn}(G))$ equals $\mathrm{Aut}(\mathcal{C}_G^+)$. By Theorem 2.7, $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$; since R is a topological isomorphism, $R(\mathrm{Inn}(G))$ must in fact be closed, and hence equal to $\mathrm{Aut}(\mathcal{C}_G^+)$. Also $\mathrm{Aut}(\mathcal{C}_G^+)$ is normal in $\mathrm{Aut}(\mathcal{C}_G)$ because $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$. Therefore

$$\mathrm{Aut}(G)/\mathrm{Inn}(G) \cong \mathrm{Aut}(\mathcal{C}_G)/\mathrm{Aut}(\mathcal{C}_G^+).$$

The statement (3) now follows by invoking Lemma 3.5. □

We use the foregoing result to show that in fact, (3) implies (2) for *any* oligomorphic group $G \leqslant_c \mathrm{Sym}(\omega)$.

Proposition 4.8. *Suppose that G is oligomorphic and $\mathrm{Out}(G)$ is profinite. Then $\mathrm{Aut}(G)$ is quasi-oligomorphic.*

Proof. We will apply Theorem 4.7 to the Borel class $\mathcal{V} = \{G\}$ and the property $P(U, G)$ saying that U is automorphic to G_a for some $a \in \omega$.

The group $\mathrm{Aut}(G)$ acts continuously on the (discrete) space of open subgroups of G: for this it suffices to verify that $\{\alpha \in \mathrm{Aut}(G) \colon \alpha(U) = U\}$ is open for each $U \leqslant_o G$, which follows from the definition of the topology on $\mathrm{Aut}(G)$ in 2.3. Therefore $\mathrm{Out}(G)$ acts continuously on the discrete space of conjugacy classes of open subgroups of G. Since $\mathrm{Out}(G)$ is profinite, every orbit of this action is finite. There are only finitely many conjugacy classes of subgroups of the form G_a. The union of the orbits of these conjugacy classes under the $\mathrm{Out}(G)$ action consists therefore of finitely many conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_m$. The property $P(U, G)$ holds precisely when $U \in \bigcup_r \mathcal{C}_r$. Thus P is Borel and there are only finitely many conjugacy classes of P-subgroups. The Assumptions 4.2 are satisfied, so $\mathrm{Aut}(G)$ is quasi-oligomorphic by (2) of Theorem 4.7. □

5. Two smooth classes

We apply the general criterion of Theorem 4.7 to two Borel classes of oligomorphic groups, thereby showing that their isomorphism relation on the class is smooth, the automorphism group of each such group is profinite, and hence its automorphism group is quasi-oligomorphic (i.e., isomorphic to an oligomorphic group).

An action of a group G on a set X corresponds to a homomorphism $G \to \mathrm{Sym}(X)$. One says that (G, X) is *faithful* if this homomorphism is 1-1, and (G, X) is *closed* if its range is closed in $\mathrm{Sym}(X)$.

5.1. Groups with no algebraicity

Let $G \leq \mathrm{Sym}(\omega)$ be oligomorphic. Recall from 3.1 that M_G denotes the canonical structure M such that $\mathrm{Aut}(M) = G$. A countable structure M has *no algebraicity* if $\mathrm{acl}(A) = A$ for each $A \subseteq M$.

Definition 5.1. G has *no algebraicity* if for each finite set $S \subseteq \omega$, the action of $G_{(S)}$ on ω has no finite 1-orbits on $\omega \setminus S$. Equivalently, the structure M_G has no algebraicity.

Note that this property depends on G as a permutation group. It is clearly a Borel property of structures with domain ω to have no algebraicity. Since the operation $G \mapsto M_G$ is Borel [23, Secton 1.3], the class of groups with no algebraicity is Borel.

In the following, let N be the substructure of M^{eq} with domain a sort $S = D/E$. We say that N is *dense* in M^{eq} if for every $b \in \mathrm{M}^{\mathrm{eq}}$ there is a finite subset $A \subseteq N$ such that b is definable from A in M^{eq}. (This property is called "regular" in [27].) Note that $\mathrm{Aut}(M)$ acts naturally on N.

Fact 5.2 (by [27, Lemma 2.10])**.** Suppose that $(\mathrm{Aut}(M), N)$ is faithful, and closed. Then N is a dense substructure of M^{eq}.

Fact 5.3 (by [27, Lemma 2.11])**.** Suppose that E is non-degenerate in the sense of Definition 3.6. Suppose that M has no algebraicity, N has no algebraicity, and N is dense in M^{eq}. Then $D = M$ and E is the identity equivalence relation.

The next result together with Theorem 4.7 yields Theorem 1.2 for the first class.

Lemma 5.4. *The hypotheses of Theorem 4.7 are satisfied in the following setting:*

- $\mathcal{V}$ *is the class of groups with no algebraicity.*
- $P(U, G)$ *holds if* $G \in \mathcal{V}$ *and* $U = G_a$ *for some* $a \in \omega$.

Proof. To verify that P is Borel, we use the standard methods surveyed in Kechris et al. [17, Section 2]. By $\mathcal{U}(S_\infty)$ they denote the Borel space of closed subgroups of S_∞, which is a subspace of the usual Effros space $\mathcal{F}(S_\infty)$. They verify that inclusion is Borel on this space. For $H \in \mathcal{U}(S_\infty)$, by T_H they denote a tree of strings over ω describing H; the map $H \mapsto T_H$ is Borel. For the Borelness of P it suffices to show that the map $\omega \times \mathcal{U}(S_\infty) \to \mathcal{U}(S_\infty)$ that sends (a, G) to G_a has Borel graph. For $H \in \mathcal{U}(S_\infty)$, we have

$$H = G_a \Leftrightarrow H \subseteq G \wedge \forall \beta \in T_H[|\beta| > a \to \beta(a) = a].$$

This condition is clearly Borel.

For each $G \in \mathcal{V}$, the number of conjugacy classes of P-subgroups equals the number of 1-orbits of G, which is finite. Clearly, the P-subgroups generate the filter of open subgroups of G. It remains to show that P is invariant for groups in $\mathcal{V}$ in the sense of Definition 4.1(1).

Claim 1. *Suppose* $G, H \in \mathcal{V}$. *For each isomorphism* $\Phi\colon G \to H$ *there is a bijection* $f\colon M_G \to M_H$ *such that* $\Phi(g) = f \circ g \circ f^{-1}$ *for each* $g \in G$.

Assuming the claim, $\Phi(G_a) = H_{f(a)}$ for each $a \in \omega$. Thus P is preserved.

We verify the claim using 5.3. Note that Φ corresponds to a bi-interpretation of M_G and M_H by [1, Cor. 1.4]. In particular, there are a sort $S = D/E$ of M_G and an onto map $f\colon D \to M_H$ with kernel E such that $\Phi = \mathrm{Aut}(f)$ in the sense of [1, 1.3]. Note that G acts on M_H via the given interpretation of M_H in M_G.

Let N be the substructure of M_G^{eq} with domain S. Since M_G and M_H are bi-interpretable, the action of $\mathrm{Aut}(M_G)$ on M_H is faithful and closed. (This is easy to verify directly, but also follows from the independently established Proposition 5.9 below.) Hence the action of G on N is faithful and closed as well.

We can assume E is non-degenerate (Definition 3.6): otherwise, say E contains the equivalence relation $E_{\{0,\ldots,k-2\},D}$ expressing that two k-tuples over D agree on the first $k-1$ elements. Let $D' = p(D)$ where p is the definable projection $M^k \to M^{k-1}$ that erases the last component. Then D' is definable, and p yields an M-definable bijection $D/E \to D'/p(E)$. So we can replace N with the structure N' that has domain $D'/p(E)$.

Clearly (G, M_H) and (G, N) are isomorphic as permutation groups. So one has algebraicity iff the other does. By hypothesis, this implies that N has no algebraicity.

By 5.3, $D = M$ and E is the identity equivalence relation. Thus $f\colon M_G \to M_H$ is a bijection. Since $\Phi = \mathrm{Aut}(f)$ in the sense of [1, p. 66], this shows $\Phi(g) = f \circ g \circ f^{-1}$, as required for the claim. □

The following fact is a well-known observation of Hrushovski; see [8, Ch. 1, Lemma 1.1.1] where it is proved in the language of permutation groups. In the subsequent remark we use the proof to show that the hypothesis that both M and N have no algebraicity is necessary in 5.3.

Fact 5.5. Let M be a countable $\aleph_0$-categorical structure. There is a countable $\aleph_0$-categorical structure N such that M and N are bi-interpretable, and N has only one 1-orbit.

Proof. Let $0 < n < \omega$ be the number of 1-orbits of M. We may replace M by the canonical structure for $\mathrm{Aut}(M)$, by naming its n-orbits with predicates of arity n. In particular, let $P_1, \dots, P_n$ be the predicates naming the 1-orbits. Let

$$\begin{aligned} D &= \bigcup_{\sigma \in S_n} \prod_{i=1}^{n} P_{\sigma(i)}, \\ E &= \{\langle \bar{x}, \bar{y} \rangle \in D^2 \colon \exists \sigma \in S_n \forall i \, [x_i = y_{\sigma(i)}]\}. \end{aligned}$$

The structure N induced by M^{eq} on D/E is what we are looking for. Notice that N is a dense subset of M^{eq}. □

Remark 5.6. In the context of the proof of 5.5, if M has more than one 1-orbit, then N has algebraicity, i.e., there is finite $A \subseteq N$ and $a \in N \setminus A$ s.t. $a \in \mathrm{acl}_N(A)$.

Proof. To save on notation, we show this for the case that M has only two distinct infinite 1-orbits, denoted P_1 and P_2. Given $\langle a, b \rangle \in D$, we denote its equivalence class by $\{a, b\}$. On N we consider the undirected graph with the edge relation

$$\{a, b\} R \{c, d\} \text{ iff } |\{a, b\} \cap \{c, d\}| = 1.$$

This relation is definable because N inherits all the definable relations from M^{eq}. Fix $x \in P_1$ and $y \in P_2$. Furthermore, fix pairwise distinct $x_i \in P_1 \setminus \{x\}$ and $y_i \in P_2 \setminus \{y\}$, for $i = 1, 2$. Define $a_i = \{x, y_i\}$ and $b_i = \{x_i, y\}$. Now $c = \{x, y\}$ is the unique vertex connected to each vertex in $A = \{a_1, a_2, b_1, b_2\}$. Thus $c \in \mathrm{dcl}_N(A) \setminus A$. □

5.2. Finitely many essential subgroups up to conjugacy

Membership of G in our second smooth class is determined solely by its lattice of open subgroups, and hence is invariant under topological isomorphism. To define this class, we need two auxiliary notions, the depth of open subgroups, and essential subgroups, Definition 5.8(3).

It is well known that for an oligomorphic group G, there are only finitely many open subgroups above any given one; this follows from 3.7(ii).

Definition 5.7. The *depth* of an open subgroup U is the minimum of all lengths of maximal chains leading from U to G. By this definition, G has depth 0.

Definition 5.8. (a) One says that an open subgroup of G is *irreducible* if it has no proper open subgroup of finite index.

(b) We say that an irreducible subgroup H of G is *almost essential* if every open subgroup of G contains a finite intersection of conjugates of H.

(c) Such a subgroup H is *essential* if it is almost essential, and its depth is minimal among the almost essential subgroups.

We provide some model theoretic context to the condition in the definition of being almost essential in 5.8. Note that it is known that an open subgroup of G is irreducible iff it is the pointwise stabiliser of some set $\mathrm{acl}(e)$ where $e \in M^{\mathrm{eq}}$.

Proposition 5.9. *Let U be an open subgroup of $G = \mathrm{Aut}(M)$. Write $U = G_e$ for $e \in M^{\mathrm{eq}}$, and let $S \subseteq M^{\mathrm{eq}}$ be the G-orbit containing e. The following are equivalent.*

(i) Every open subgroup of G contains a finite intersection of conjugates of U.

(ii) The permutation group induced by G on S is faithful and closed.

(iii) The structure induced by M^{eq} on S is bi-interpretable with M.

Proof. (i) $\to$ (ii). Firstly, the kernel of the map r in (ii) is contained in every open subgroup by (i), and so must be trivial. Secondly, for every $a \in M$ there is a finite $B \subseteq S$ such that $a \in \mathrm{dcl}_{M^{\mathrm{eq}}}(B)$. This implies that the group $r(G)$ induced on S is closed: if $r(g_n)$ $(n \in \omega)$ is a convergent sequence in the range of r, then the g_n converge and the limit g is in G; so the $r(g_n)$ converge to $r(g)$.

(ii) $\to$ (i). r gives a continuous isomorphism between the Polish groups G and $r(G)$, so it is a homeomorphism. Thus, if $H \leqslant G$ is open, then $r(H)$ is open and so contains $G_{(B)}$ for some finite subset B of S.

(iii) $\to$ (ii) follows from the above-mentioned correspondence of topological isomorphisms and bi-interpretations [1].
(ii) $\to$ (iii). The structure N induced by M^{eq} on S is bi-definable with the structure obtained by naming all the orbits of the action of G on S. So, if the map $r : G \to \mathrm{Sym}(S)$ is injective and has closed range, then $\mathrm{Aut}(N) \cong G$. □

We are now ready to prove the statements of the introduction on groups with finitely many conjugacy classes of essential subgroups.

Lemma 5.10. *The hypotheses of Theorem 4.7 are satisfied in the following setting:*

- $\mathcal{V}$ *is the class of oligomorphic groups with finitely many essential subgroups up to conjugacy;*
- $P(U, G)$ *holds if* $G \in \mathcal{V}$ *and* U *is an essential subgroup.*

Proof. Let $P_0(U, G)$ express that U is an essential subgroup of an oligomorphic group G. To show that the binary relation P_0 is Borel, we use the notation from the proof of 4.7(1), in particular the Borel function $G \mapsto \mathcal{W}(G)$ and the associated assignment $G \mapsto \nu_G$ where ν_G is a bijection $\omega \to \mathcal{W}(G)$. There is an $L_{\omega_1,\omega}$-formula ϕ with one free variable expressing in $\mathcal{W}(G)$ that a subgroup is essential; indeed, the three conditions in Definition 5.8(a)–(c) are expressed by quantifying over finite collections of objects. Given an open subgroup U of the oligomorphic group G, we have that U is essential in G iff $\mathcal{W}(G) \models \phi(U)$; this condition is Borel using the identification of ω and $\mathcal{W}(G)$ uniformly provided by the maps ν_G.

Next we show that the class $\mathcal{V}$ is Borel. As above, since P_0 is a Borel property, the map $G \mapsto \mathcal{W}^{P_0}(G)$ is Borel, where $\mathcal{W}^{P_0}(G)$ is $\mathcal{W}(G)$ extended by a unary predicate saying that a subgroup satisfies P_0. Conjugacy of subgroups U, V can be expressed in $\mathcal{W}(G)$ via the formula $\exists A[U = A \cdot A^{-1} \wedge V = A^{-1} \cdot A]$. So there is another $L_{\omega_1,\omega}$-sentence saying that there are finitely many conjugacy classes of essential subgroups.

P is clearly invariant and sub-basic in the sense of Definition 4.1. □

Recall from the introduction that an oligomorphic group is $\mathfrak{G}$-*finite* if each open subgroup contains a least open subgroup of finite index. Borelness of this class can be shown in a similar way as above; we omit the proof.

Proposition 5.11. *Let* G *be* $\mathfrak{G}$*-finite. Then* G *has an essential subgroup.*

Proof. By 5.5 we can assume that $G = \mathrm{Aut}(M)$ and G has a single 1-orbit. For each $a \in M$, let $H(a) \leqslant G_a$ be the smallest open subgroup of G that is a subgroup of G_a with finite index. Note that $H(a)$ is irreducible in the sense

of Definition 5.8(a). Since each open subgroup of G contains the pointwise stabiliser of a finite set, each group of the form $H(a)$ is almost essential in the sense of 5.8. Hence there is an almost essential subgroup of minimal depth, which is then essential by definition. □

5.3. Smoothness for weak elimination of imaginaries

The following is equivalent to the standard definition of weak elimination of imaginaries; see e.g. [10, Lemma 1.3] for a definition as well as a proof of the equivalence.

Definition 5.12. Let M be $\aleph_0$-categorical. One says that M has *weak elimination of imaginaries (w.e.i.)* if for every open $U \leqslant \mathrm{Aut}(M)$ there is a unique finite algebraically closed set $A \subseteq M$ such that $G_{(A)} \leqslant U \leqslant G_{\{A\}}$.

In the following we reprove the main result of [26, Th 1.1] in our framework (relying on some technical lemmas in [26]), and use this to obtain new information about the groups $\mathrm{Aut}(G)$ where $G = \mathrm{Aut}(M)$ and M has w.e.i.

Corollary 5.13 (to Theorem 4.7)**.** *The class $\mathcal{C}$ of automorphism groups of $\aleph_0$-categorical structures that have w.e.i. is smooth [26]. Furthermore, if M has w.e.i. and $G = \mathrm{Aut}(M)$, then $\mathrm{Aut}(G)$ is quasi-oligomorphic and $\mathrm{Out}(G)$ is profinite.*

Proof. $\mathcal{C}$ is Borel by [26, 2.18]. It suffices to show that $\mathcal{C}$ is a subclass of the class $\mathcal{V}$ from Lemma 5.10. Let $G = \mathrm{Aut}(M)$ where M has w.e.i.

To verify that G has an essential subgroup, by 5.11 it suffices to show that G is $\mathfrak{G}$-finite. To see this, suppose a finite set $A \subseteq M$ is algebraically closed. We claim that if $V \subseteq G_{(A)}$ is open and of finite index in $G_{(A)}$ then $V = G_{(A)}$. We have $V = G_e$ for some imaginary e. Let $B = \mathrm{acl}(e) \cap M$. Since M has w.e.i. we have $G_{(B)} \leqslant G_{(e)} \leqslant G_{\{B\}}$. So $A \subseteq B$ (as $G_{(B)} \leqslant G_{(A)}$) and $e \in \mathrm{acl}(A)$ (as $G_{(e)}$ is of finite index in $G_{(A)}$). It follows that $A = B$.

The irreducible subgroups of G correspond precisely to the pointwise stabilisers of finite algebraically closed subsets of M by [26, 2.11]. For any fixed depth $k < \omega$ there are only finitely many conjugacy classes of such stabilisers by [26, 2.17]. A fortiori, there are only finitely many conjugacy classes of essential subgroups. □

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